From Encyclopedia of Mathematics Hermitian-symmetric matrix, self-conjugate matrix A square matrix $ A = \| a _ {ik} \| $ over $ \mathbf C $ that is the same as its Hermitian-conjugate matrix $$ A ^ {*} = \overline{A}\; {} ^ {T} = \| \overline{ {a _ {ki} }}\; \| , $$ that is, a matrix whose entries satisfy the condition $ a _ {ik} = \overline{ {a _ {ki} }}\; $. If all the $ a _ {ik} \in \mathbf R $, then a Hermitian matrix is symmetric (cf. Symmetric matrix). The Hermitian matrices of a fixed order form a vector space over $ \mathbf R $. If $ A $ and $ B $ are two Hermitian matrices of the same order, then so is $ AB + BA $. Under the operation $ A \cdot B = (AB + BA) / 2 $ the Hermitian matrices (of order $ n $) form a Jordan algebra. The product $ AB $ of two Hermitian matrices is itself Hermitian if and only if $ A $ and $ B $ commute. The Hermitian matrices of order $ n $ are the matrices of Hermitian transformations of an $ n $- dimensional unitary space in an orthonormal basis (see Self-adjoint linear transformation). On the other hand, Hermitian matrices are the matrices of Hermitian forms in an $ n $- dimensional complex vector space. Like Hermitian forms (cf. Hermitian form), Hermitian matrices can be defined over any skew-field with an anti-involution. All eigen values of a Hermitian matrix are real. For every Hermitian matrix $ A $ there exists a unitary matrix $ U $ such that $ U ^ {-} 1 AU $ is a real diagonal matrix. A Hermitian matrix is called non-negative (or positive semi-definite) if all its principal minors are non-negative, and positive definite if they are all positive. Non-negative (positive-definite) Hermitian matrices correspond to non-negative (positive-definite) Hermitian linear transformations and Hermitian forms. References [a1] F.R. [F.R. Gantmakher] Gantmacher, "Matrix theory" , 1–2 , Chelsea, reprint (1959) (Translated from Russian)[a2] B. Noble, J.W. Daniel, "Applied linear algebra" , Prentice-Hall (1979) How to Cite This Entry: Hermitian matrix. Encyclopedia of Mathematics. URL: article was adapted from an original article by A.L. Onishchik (originator), which appeared in Encyclopedia of Mathematics - ISBN 1402006098. See original article Matrix equal to its conjugate-transpose For matrices with symmetry over the real number field, see Symmetric matrix. In mathematics, a Hermitian matrix (or self-adjoint matrix) is a complex square matrix that is equal to its own conjugate transpose—that is, the element in the i-th row and j-th column is equal to the complex conjugate of the element in the j-th row and i-th column, for all indices i and j: A is Hermitian ⟺ a i j = a j i ¯ {\displaystyle A{\text{ is Hermitian}}\quad \iff \quad a_{ij}={\overline {a_{ji}}}} or in matrix form: A is Hermitian ⟺ A = A T ¯ . {\displaystyle A{\text{ is Hermitian}}\quad \iff \quad A={\overline {A^{\mathsf {T}}}}.} Hermitian matrices can be understood as the complex extension of real symmetric matrices. If the conjugate transpose of a matrix A {\displaystyle A} is denoted by A H , {\displaystyle A^{\mathsf {H}},} then the Hermitian property can be written concisely as A is Hermitian ⟺ A = A H {\displaystyle A{\text{ is Hermitian}}\quad \iff \quad A=A^{\mathsf {H}}} Hermitian matrices are named after Charles Hermite,[1] who demonstrated in 1855 that matrices of this form share a property with real symmetric matrices of always having real eigenvalues. Other, equivalent notations in common use are A H = A † = A ∗ , {\displaystyle A^{\mathsf {H}}=A^{\dagger }=A^{\ast },} although in quantum mechanics, A ∗ {\displaystyle A^{\ast }} typically means the complex conjugate only, and not the conjugate transpose. Hermitian matrices can be characterized in a number of equivalent ways, some of which are listed below: A square matrix A {\displaystyle A} is Hermitian if and only if it is equal to its conjugate transpose, that is, it satisfies ⟨ w , A v ⟩ = ⟨ A w , v ⟩ , {\displaystyle \langle \mathbf {w} ,A\mathbf {v} \rangle =\langle A\mathbf {w} ,\mathbf {v} \rangle ,} for any pair of vectors v , w , {\displaystyle \mathbf {v} ,\mathbf {w} ,} where ⟨ ⋅ , ⋅ ⟩ {\displaystyle \langle \cdot ,\cdot \rangle } denotes the inner product operation. This is also the way that the more general concept of self-adjoint operator is defined. An n × n {\displaystyle n\times {}n} matrix A {\displaystyle A} is Hermitian if and only if ⟨ v , A v ⟩ ∈ R , for all v ∈ C n . {\displaystyle \langle \mathbf {v} ,A\mathbf {v} \rangle \in \mathbb {R} ,\quad {\text{for all }}\mathbf {v} \in \mathbb {C} ^{n}.} A square matrix A {\displaystyle A} is Hermitian if and only if it is unitarily diagonalizable with real eigenvalues. Hermitian matrices are fundamental to quantum mechanics because they describe operators with necessarily real eigenvalues. An eigenvalue a {\displaystyle a} of an operator A ^ {\displaystyle {\hat {A}}} on some quantum state | ψ ⟩ {\displaystyle |\psi \rangle } is one of the possible measurement outcomes of the operator, which requires the operators to have real eigenvalues. In signal processing, Hermitian matrices are utilized in tasks like Fourier analysis and signal representation.[2] The eigenvalues and eigenvectors of Hermitian matrices play a crucial role in analyzing signals and extracting meaningful information. Hermitian matrices are extensively studied in linear algebra and numerical analysis. They have well-defined spectral properties, and many numerical algorithms, such as the Lanczos algorithm, exploit these properties for efficient computations. Hermitian matrices also appear in techniques like singular value decomposition (SVD) and eigenvalue decomposition. In statistics and machine learning, Hermitian matrices are used in covariance matrices, where they represent the relationships between different variables. The positive definiteness of a Hermitian covariance matrix ensures the well-definedness of multivariate distributions.[3] Hermitian matrices are applied in the design and analysis of communications system, especially in the field of multiple-input multiple-output (MIMO) systems. Channel matrices in MIMO systems often exhibit Hermitian properties. In graph theory, Hermitian matrices are used to study the spectra of graphs. The Hermitian Laplacian matrix is a key tool in this context, as it is used to analyze the spectra of mixed graphs.[4] The Hermitian-adjacency matrix of a mixed graph is another important concept, as it is a Hermitian matrix that plays a role in studying the energies of mixed graphs.[5] In this section, the conjugate transpose of matrix A {\displaystyle A} is denoted as A H , {\displaystyle A^{\mathsf {H}},} the transpose of matrix A {\displaystyle A} is denoted as A T {\displaystyle A^{\mathsf {T}}} and conjugate of matrix A {\displaystyle A} is denoted as A ¯ . {\displaystyle {\overline {A}}.} See the following example: [0 a − i b c − i d a + i b 1 m − i n c + i d m + i n 2] {\displaystyle {\begin{bmatrix}0&a-ib&c-id\\a+ib&1&m-in\\c+id&m+in&2\end{bmatrix}}} The diagonal elements must be real, as they must be their own complex conjugate. Well-known families of Hermitian matrices include the Pauli matrices, the Gell-Mann matrices and their generalizations. In theoretical physics such Hermitian matrices are often multiplied by imaginary coefficients,[6][7] which results in skew-Hermitian matrices. Here, we offer another useful Hermitian matrix using an abstract example. If a square matrix A {\displaystyle A} equals the product of a matrix with its conjugate transpose, that is, A = B B H , {\displaystyle A=BB^{\mathsf {H}},} then A {\displaystyle A} is a Hermitian positive semi-definite matrix. Furthermore, if B {\displaystyle B} is row full-rank, then A {\displaystyle A} is positive definite. The entries on the main diagonal (top left to bottom right) of any Hermitian matrix are real. Proof By definition of the Hermitian matrix H i j = H ¯ j i {\displaystyle H_{ij}={\overline {H}}_{ji}} so for i = j the above follows, as a number can equal its complex conjugate only if the imaginary parts are zero. Only the main diagonal entries are necessarily real; Hermitian matrices can have arbitrary complex-valued entries in their off-diagonal elements, as long as diagonally-opposite entries are complex conjugates. A matrix that has only real entries is symmetric if and only if it is a Hermitian matrix. A real and symmetric matrix is simply a special case of a Hermitian matrix. Proof H i j = H ¯ j i {\displaystyle H_{ij}={\overline {H}}_{ji}} by definition. Thus H i j = H j i {\displaystyle H_{ij}=H_{ji}} (matrix symmetry) if and only if H i j = H ¯ i j {\displaystyle H_{ij}={\overline {H}}_{ij}} (H i j {\displaystyle H_{ij}} is real). So, if a real anti-symmetric matrix is multiplied by a real multiple of the imaginary unit i , {\displaystyle i,} then it becomes Hermitian. Every Hermitian matrix is a normal matrix. That is to say, A A H = A H A . {\displaystyle AA^{\mathsf {H}}=A^{\mathsf {H}}A.} Proof A = A H , {\displaystyle A=A^{\mathsf {H}},} so A A H = A A = A H A . {\displaystyle AA^{\mathsf {H}}=AA=A^{\mathsf {H}}A.} The finite-dimensional spectral theorem says that any Hermitian matrix can be diagonalized by a unitary matrix, and that the resulting diagonal matrix has only real entries. This implies that all eigenvalues of a Hermitian matrix A with dimension n are real, and that A has n linearly independent eigenvectors. Moreover, a Hermitian matrix has orthogonal eigenvectors for distinct eigenvalues. Even if there are degenerate eigenvalues, it is always possible to find an orthogonal basis of Cn consisting of n eigenvectors of A. The sum of any two Hermitian matrices is Hermitian. Proof (A + B) i j = A i j + B i j = A ¯ j i + B ¯ j i = (A + B) ¯ j i , {\displaystyle (A+B)_{ij}=A_{ij}+B_{ij}={\overline {A}}_{ji}+{\overline {B}}_{ji}={\overline {(A+B)}}_{ji},} as claimed. The inverse of an invertible Hermitian matrix is Hermitian as well. Proof If A − 1 A = I , {\displaystyle A^{-1}A=I,} then I = I H = (A − 1 A) H = A H (A − 1) H = A (A − 1) H , {\displaystyle I=I^{\mathsf {H}}=\left(A^{-1}A\right)^{\mathsf {H}}=A^{\mathsf {H}}\left(A^{-1}\right)^{\mathsf {H}}=A\left(A^{-1}\right)^{\mathsf {H}},} so A − 1 = (A − 1) H {\displaystyle A^{-1}=\left(A^{-1}\right)^{\mathsf {H}}} as claimed. The product of two Hermitian matrices A and B is Hermitian if and only if AB = BA. Proof (A B) H = (A B) T ¯ = B T A T ¯ = B T ¯   A T ¯ = B H A H = B A . {\displaystyle (AB)^{\mathsf {H}}={\overline {(AB)^{\mathsf {T}}}}={\overline {B^{\mathsf {T}}A^{\mathsf {T}}}}={\overline {B^{\mathsf {T}}}}\ {\overline {A^{\mathsf {T}}}}=B^{\mathsf {H}}A^{\mathsf {H}}=BA.} Thus (A B) H = A B {\displaystyle (AB)^{\mathsf {H}}=AB} if and only if A B = B A . {\displaystyle AB=BA.} Thus An is Hermitian if A is Hermitian and n is an integer. If A and B are Hermitian, then ABA is also Hermitian. Proof (A B A) H = (A (B A)) H = (B A) H A H = A H B H A H = A B A {\displaystyle (ABA)^{\mathsf {H}}=(A(BA))^{\mathsf {H}}=(BA)^{\mathsf {H}}A^{\mathsf {H}}=A^{\mathsf {H}}B^{\mathsf {H}}A^{\mathsf {H}}=ABA} For an arbitrary complex valued vector v the product v H A v {\displaystyle \mathbf {v} ^{\mathsf {H}}A\mathbf {v} } is real because of v H A v = (v H A v) H . {\displaystyle \mathbf {v} ^{\mathsf {H}}A\mathbf {v} =\left(\mathbf {v} ^{\mathsf {H}}A\mathbf {v} \right)^{\mathsf {H}}.} This is especially important in quantum physics where Hermitian matrices are operators that measure properties of a system, e.g. total spin, which have to be real. The Hermitian complex n-by-n matrices do not form a vector space over the complex numbers, ℂ, since the identity matrix In is Hermitian, but i In is not. However the complex Hermitian matrices do form a vector space over the real numbers ℝ. In the 2n2-dimensional vector space of complex n × n matrices over ℝ, the complex Hermitian matrices form a subspace of dimension n2. If Ejk denotes the n-by-n matrix with a 1 in the j,k position and zeros elsewhere, a basis (orthonormal with respect to the Frobenius inner product) can be described as follows: E j j for 1 ≤ j ≤ n (n matrices) {\displaystyle E_{jj}{\text{ for }}1\leq j\leq n\quad (n{\text{ matrices}})} together with the set of matrices of the form 1 2 (E j k + E k j) for 1 ≤ j < k ≤ n (n 2 − n 2 matrices) {\displaystyle {\frac {1}{\sqrt {2}}}\left(E_{jk}+E_{kj}\right){\text{ for }}1\leq j