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Tests for series convergence and divergence

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If the series is convergent determine the value of the series. \(\displaystyle {s_n} = \frac{{5 + 8{n^2}}}{{2 - 7{n^2}}}\) Solution \(\displaystyle {s_n} = \frac{{{n^2}}}{{5 + 2n}}\) Solution For problems 5 & 6 show that the series is divergent. \(\displaystyle \sum\limits_{n = 0}^\infty {\frac{{3n\,{{\bf{e}}^n}}}{{{n^2} + 1}}} \) Solution \(\displaystyle \sum\limits_{n = 5}^\infty {\frac{{6 + 8n + 9{n^2}}}{{3 + 2n + {n^2}}}} \) Solution Learning Outcomes Describe a strategy for testing the convergence of a given series At this point, we have a long list of convergence tests. However, not all tests can be used for all series. When given a series, we must determine which test is the best to use. Here is a strategy for finding the best test to apply. Consider a series [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex]. In the steps below, we outline a strategy for determining whether the series converges. Is [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex] a familiar series? For example, is it the harmonic series (which diverges) or the alternating harmonic series (which converges)? Is it a [latex]p-\text{series}[/latex] or geometric series? If so, check the power [latex]p[/latex] or the ratio [latex]r[/latex] to determine if the series converges. Is it an alternating series? Are we interested in absolute convergence or just convergence? If we are just interested in whether the series converges, apply the alternating series test. If we are interested in absolute convergence, proceed to step [latex]3[/latex], considering the series of absolute values [latex]\displaystyle\sum _{n=1}^{\infty }|{a}_{n}|[/latex]. Is the series similar to a [latex]p-\text{series}[/latex] or geometric series? If so, try the comparison test or limit comparison test. Do the terms in the series contain a factorial or power? If the terms are powers such that [latex]{a}_{n}={b}_{n}^{n}[/latex], try the root test first. Otherwise, try the ratio test first. Use the divergence test. If this test does not provide any information, try the integral test. For each of the following series, determine which convergence test is the best to use and explain why. Then determine if the series converges or diverges. If the series is an alternating series, determine whether it converges absolutely, converges conditionally, or diverges. [latex]\displaystyle\sum _{n=1}^{\infty }\frac{{n}^{2}+2n}{{n}^{3}+3{n}^{2}+1}[/latex] [latex]\displaystyle\sum _{n=1}^{\infty }\frac{{\left(-1\right)}^{n+1}\left(3n+1\right)}{n\text{!}}[/latex] [latex]\displaystyle\sum _{n=1}^{\infty }\frac{{e}^{n}}{{n}^{3}}[/latex] [latex]\displaystyle\sum _{n=1}^{\infty }\frac{{3}^{n}}{{\left(n+1\right)}^{n}}[/latex] For the series [latex]\displaystyle\sum _{n=1}^{\infty }\frac{{2}^{n}}{{3}^{n}+n}[/latex], determine which convergence test is the best to use and explain why. Watch the following video to see the worked solution to the above Try IT. For closed captioning, open the video on its original page by clicking the Youtube logo in the lower right-hand corner of the video display. In YouTube, the video will begin at the same starting point as this clip, but will continue playing until the very end. You can view the transcript for this segmented clip of “Mixed Convergence Tests” here (opens in new window). In the following table, we summarize the convergence tests and when each can be applied. Note that while the comparison test, limit comparison test, and integral test require the series [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex] to have nonnegative terms, if [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex] has negative terms, these tests can be applied to [latex]\displaystyle\sum _{n=1}^{\infty }|{a}_{n}|[/latex] to test for absolute convergence. Summary of Convergence Tests Series or Test Conclusions Comments Divergence Test For any series [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex], evaluate [latex]\underset{n\to \infty }{\text{lim}}{a}_{n}[/latex]. If [latex]\underset{n\to \infty }{\text{lim}}{a}_{n}=0[/latex], the test is inconclusive. This test cannot prove convergence of a series. If [latex]\underset{n\to \infty }{\text{lim}}{a}_{n}≠ 0[/latex], the series diverges. Geometric Series [latex]\displaystyle\sum _{n=1}^{\infty }a{r}^{n - 1}[/latex] If [latex]|r| [latex]\frac{a}{\left(1-r\right)}[/latex]. Any geometric series can be reindexed to be written in the form [latex]a+ar+a{r}^{2}+\cdots[/latex], where [latex]a[/latex] is the initial term and [latex]r[/latex] is the ratio. If [latex]|r|≥ 1[/latex], the series diverges. p-Series [latex]\displaystyle\sum _{n=1}^{\infty }\frac{1}{{n}^{p}}[/latex] If [latex]p>1[/latex], the series converges. For [latex]p=1[/latex], we have the harmonic series [latex]\displaystyle\sum _{n=1}^{\infty }\frac{1}{n}[/latex]. If [latex]p≤ 1[/latex], the series diverges. Comparison Test For [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex] with nonnegative terms, compare with a known series [latex]\displaystyle\sum _{n=1}^{\infty }{b}_{n}[/latex]. If [latex]{a}_{n}≤ {b}_{n}[/latex] for all [latex]n≥ N[/latex] and [latex]\displaystyle\sum _{n=1}^{\infty }{b}_{n}[/latex] converges, then [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex] converges. Typically used for a series similar to a geometric or [latex]p[/latex] -series. It can sometimes be difficult to find an appropriate series. If [latex]{a}_{n}≥ {b}_{n}[/latex] for all [latex]n≥ N[/latex] and [latex]\displaystyle\sum _{n=1}^{\infty }{b}_{n}[/latex] diverges, then [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex] diverges. Limit Comparison Test For [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex] with positive terms, compare with a series [latex]\displaystyle\sum _{n=1}^{\infty }{b}_{n}[/latex] by evaluating [latex]L=\underset{n\to \infty }{\text{lim}}\frac{{a}_{n}}{{b}_{n}}[/latex]. If [latex]L[/latex] is a real number and [latex]L≥ 0[/latex], then [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex] and [latex]\displaystyle\sum _{n=1}^{\infty }{b}_{n}[/latex] both converge or both diverge. Typically used for a series similar to a geometric or [latex]p[/latex] -series. Often easier to apply than the comparison test. If [latex]L=0[/latex] and [latex]\displaystyle\sum _{n=1}^{\infty }{b}_{n}[/latex] converges, then [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex] converges. If [latex]L=\infty[/latex] and [latex]\displaystyle\sum _{n=1}^{\infty }{b}_{n}[/latex] diverges, then [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex] diverges. Integral Test If there exists a positive, continuous, decreasing function [latex]f[/latex] such that [latex]{a}_{n}=f\left(n\right)[/latex] for all [latex]n≥ N[/latex], evaluate [latex]{\displaystyle\int }_{N}^{\infty }f\left(x\right)dx[/latex]. [latex]{\displaystyle\int }_{N}^{\infty }f\left(x\right)dx[/latex] and [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex] both converge or both diverge. Limited to those series for which the corresponding function [latex]f[/latex] can be easily integrated. Alternating Series [latex]\displaystyle\sum _{n=1}^{\infty }{\left(-1\right)}^{n+1}{b}_{n}\text{or}\displaystyle\sum _{n=1}^{\infty }{\left(-1\right)}^{n}{b}_{n}[/latex] If [latex]{b}_{n+1}≤ {b}_{n}[/latex] for all [latex]n≥ 1[/latex] and [latex]{b}_{n}\to 0[/latex], then the series converges. Only applies to alternating series. Ratio Test For any series [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex] with nonzero terms, let [latex]\rho =\underset{n\to \infty }{\text{lim}}|\frac{{a}_{n+1}}{{a}_{n}}|[/latex]. If [latex]0≤ \rho Often used for series involving factorials or exponentials. If [latex]\rho >1\text{or}\rho =\infty[/latex], the series diverges. If [latex]\rho =1[/latex], the test is inconclusive. Root Test For any series [latex]\displaystyle\sum _{n=1}^{\infty }{a}_{n}[/latex], let [latex]\rho =\underset{n\to \infty }{\text{lim}}\sqrt[n]{|{a}_{n}|}[/latex]. If [latex]0≤ \rho Often used for series where [latex]|{a}_{n}|=={b}_{n}^{n}[/latex]. If [latex]\rho >1\text{or}\rho =\infty[/latex], the series diverges. If [latex]\rho =1[/latex], the test is inconclusive. Dozens of series exist that converge to [latex]\pi[/latex] or an algebraic expression containing [latex]\pi[/latex]. Here we look at several examples and compare their rates of convergence. By rate of convergence, we mean the number of terms necessary for a partial sum to be within a certain amount of the actual value. The series representations of [latex]\pi[/latex] in the first two examples can be explained using Maclaurin series, which are discussed in the next chapter. The third example relies on material beyond the scope of this text. The series [latex]\pi =4\displaystyle\sum _{n=1}^{\infty }\frac{{\left(-1\right)}^{n+1}}{2n - 1}=4-\frac{4}{3}+\frac{4}{5}-\frac{4}{7}+\frac{4}{9}-\cdots[/latex] was discovered by Gregory and Leibniz in the late [latex]1600\text{s}\text{.}[/latex] This result follows from the Maclaurin series for [latex]f\left(x\right)=\{\tan}^{-1}x[/latex]. We will discuss this series in the next chapter. Prove that this series converges. Evaluate the partial sums [latex]{S}_{n}[/latex] for [latex]n=10,20,50,100[/latex]. Use the remainder estimate for alternating series to get a bound on the error [latex]{R}_{n}[/latex]. What is the smallest value of [latex]N[/latex] that guarantees [latex]|{R}_{N}| The series [latex]\begin{array}{cc}\hfill \pi & =6 {\displaystyle\sum _{n=0}^{\infty }}\frac{\left(2n\right)\text{!}}{{2}^{4n+1}{\left(n\text{!}\right)}^{2}\left(2n+1\right)}\hfill \\ & =6\left(\frac{1}{2}+\frac{1}{2\cdot 3}{\left(\frac{1}{2}\right)}^{3}+\frac{1\cdot 3}{2\cdot 4\cdot 5}\cdot {\left(\frac{1}{2}\right)}^{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}{\left(\frac{1}{2}\right)}^{7}+\cdots \right)\hfill \end{array}[/latex] has been attributed to Newton in the late [latex]1600\text{s}\text{.}[/latex] The proof of this result uses the Maclaurin series for [latex]f\left(x\right)={\sin}^{-1}x[/latex]. Prove that the series converges. Evaluate the partial sums [latex]{S}_{n}[/latex] for [latex]n=5,10,20[/latex]. Compare [latex]{S}_{n}[/latex] to [latex]\pi[/latex] for [latex]n=5,10,20[/latex] and discuss the number of correct decimal places. The series [latex]\frac{1}{\pi }=\frac{\sqrt{8}}{9801}\displaystyle\sum _{n=0}^{\infty }\frac{\left(4n\right)\text{!}\left(1103+26390n\right)}{{\left(n\text{!}\right)}^{4}{396}^{4n}}[/latex] was discovered by Ramanujan in the early [latex]1900\text{s}\text{.}[/latex] William Gosper, Jr., used this series to calculate [latex]\pi[/latex] to an accuracy of more than [latex]17[/latex] million digits in the [latex]\text{mid-}1980\text{s}\text{.}[/latex] At the time, that was a world record. Since that time, this series and others by Ramanujan have led mathematicians to find many other series representations for [latex]\pi[/latex] and [latex]\frac{1}{\pi}[/latex]. Prove that this series converges. Evaluate the first term in this series. Compare this number with the value of [latex]\pi[/latex] from a calculating utility. To how many decimal places do these two numbers agree? What if we add the first two terms in the series? Investigate the life of Srinivasa Ramanujan [latex]\left(1887\text{-}1920\right)[/latex] and write a brief summary. Ramanujan is one of the most fascinating stories in the history of mathematics. He was basically self-taught, with no formal training in mathematics, yet he contributed in highly original ways to many advanced areas of mathematics. In mathematics, convergence tests are essential tools used to determine whether an infinite series converges or diverges. Convergence Tests is a compilation of many tests which help us find the convergence of infinite series. Infinite series are sums of infinitely many terms, and understanding their behavior is crucial in various fields of science and engineering. There are several convergence tests, each designed to handle different types of series.One of the fundamental tests is the nth Term Test, which states that if the limit of the nth term of a series does not approach zero, the series diverges. However, if the limit does approach zero, the test is inconclusive, and other tests must be used.In this article, we will discuss various tests to check the convergence of the series.What are Convergence Tests?Convergence tests are mathematical tools used to determine whether an infinite series converges or diverges.An infinite series is an expression of the form \sum_{n=1}^{\infty} a_n, where a represents the terms of the series. Convergence of a series implies that as more terms are added, the series approaches a finite value. Conversely, divergence implies that the series grows without bound or oscillates as more terms are added.Types of Convergence TestsSome of the common test to check convergence of any series are:Integral TestRatio TestComparison TestRoot Test (Cauchy’s Root Test)Alternating Series Test (Leibniz’s Test)Raabe’s TestIntegral TestIntegral Test is used to determine the convergence or divergence of an infinite series by comparing it to an improper integral. Consider a series \sum_{n=1}^{\infty}a_n, where an = f(n) and f(x) is a continuous, positive, and decreasing function for x ≥ 1. The Integral Test states that:If \int_{1}^{\infty} f(x) \, dx converges, then the series \sum_{n=1}^{\infty} a_n converges.If \int_{1}^{\infty} f(x) \, dx diverges, then the series \sum_{n=1}^{\infty} a_n diverges.Example: Consider the series \sum_{n=1}^{\infty} \frac{1}{n^p}. Solution:Let f(x) = 1/xp.Check if f(x) is continuous, positive, and decreasing for x ≥ 1:f(x) = 1/xp is continuous for x ≥ 1 when p > 0.f(x) > 0 for x > 0 when p > 0.f(x) is decreasing for x ≥ 1 when p > 0.Now, apply the Integral Test: \int_{1}^{\infty} \frac{1}{x^p} \, dxEvaluate the integral: \int_{1}^{\infty} \frac{1}{x^p} \, dx = \left(\frac{x^{1-p}}{1-p} \right|_{1}^{\infty}If p > 1, the integral converges because: \left. \frac{x^{1-p}}{1-p} \right|_{1}^{\infty} = \frac{1}{1-p} \lim_{x \to \infty} \frac{1}{x^{p-1}} = 0.Thus, \sum_{n=1}^{\infty} \frac{1}{n^p} converges.If p ≤ 1, the integral diverges because: \left. \frac{x^{1-p}}{1-p} \right|_{1}^{\infty} \text{ diverges for } p \leq 1 diverges for p ≤ 1Thus, \sum_{n=1}^{\infty} \frac{1}{n^p} diverges.Ratio TestRatio Test provides a straightforward way to determine the convergence of an infinite series by examining the ratio of successive terms. Consider a series \sum_{n=1}^{\infty} a_n.To apply the Ratio Test, we define the ratio L as follows:L = \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right|The Ratio Test then gives us the following criteria:If L < 1, the series \sum_{n=1}^{\infty} a_n converges absolutely.If L > 1 or L = ∞, the series \sum_{n=1}^{\infty} a_n diverges.If L = 1, the test is inconclusive, and the series may converge or diverge; other methods must be used to determine the behavior of the series.Example: Consider the series \sum_{n=1}^{\infty} \frac{n!}{n^n}.Solution:Identify a_n = \frac{n!}{n^n}.Calculate the ratio of successive terms: \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right|Simplify the expression: \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{(n+1) \cdot n!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right| = \left| \frac{n}{(n+1)^n} \cdot \frac{1}{n+1} \right|Further simplification: \left| \frac{a_{n+1}}{a_n} \right| = \left| \left(\frac{n}{n+1} \right)^n \cdot \frac{1}{n+1} \right|Take the limit as n → ∞: L = \lim_{n \to \infty} \left| \left(\frac{n}{n+1} \right)^n \cdot \frac{1}{n+1} \right|Recognize that \left(\frac{n}{n+1} \right)^n \approx e^{-1} as n → ∞: L = \lim_{n \to \infty} \left(e^{-1} \cdot \frac{1}{n+1} \right) = 0Since L = 0 < 1, the series \sum_{n=1}^{\infty} \frac{n!}{n^n} converges absolutely.Comparison TestConsider two series \sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n, where an, bn > 0 for all n.Direct Comparison TestIf 0 ≤ an ≤ bn for all n and \sum_{n=1}^{\infty} b_n converges, then \sum_{n=1}^{\infty} a_n also converges.If 0 ≤ bn ≤ an for all n and \sum_{n=1}^{\infty} b_n diverges, then \sum_{n=1}^{\infty} a_n also diverges.Limit Comparison TestIf an > 0 and bn > 0 for all n and the limitL = \lim_{n \to \infty} \frac{a_n}{b_n}Exists and is a positive finite number (i.e., 0 < L < ∞), then either both series \sum_{n=1}^{\infty} a_n \text{ and } \sum_{n=1}^{\infty} b_n converge or both diverge.Example (Direct Comparison Test): Consider the series \sum_{n=1}^{\infty} \frac{1}{n^2 + 1}.Solution:Choose a comparison series \sum_{n=1}^{\infty} b_n such that 0 ≤ an ≤ bn.Let bn = 1/n2.Note that \frac{1}{n^2 + 1} \leq \frac{1}{n^2} for all n ≥ 1.Since, \sum_{n=1}^{\infty} \frac{1}{n^2} is a p-series with p = 2 > 1, it converges. By the Direct Comparison Test, \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} also converges.Example (Limit Comparison Test): Consider the series \sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}.Solution:Choose a comparison series \sum_{n=1}^{\infty} b_n.Let bn = 1/n.Compute the limit: \lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{\frac{n^2}{n^3 + 1}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^3}{n^3 + 1} = \lim_{n \to \infty} \frac{n^3}{n^3(1 + \frac{1}{n^3})} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n^3}} = 1Since the limit is a positive finite number, and \sum_{n=1}^{\infty} \frac{1}{n} (harmonic series) diverges, the series \sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1} also diverges by the Limit Comparison Test.Root Test (Cauchy’s Root Test)Root Test, also known as Cauchy’s Root Test, is a method used to determine the convergence or divergence of an infinite series by examining the n-th root of the absolute value of the terms of the series.For a series \sum_{n=1}^{\infty} a_n, define L as: L = \lim_{n \to \infty} \sqrt[n]{|a_n|}The conclusions of the Root Test are as follows:If L < 1, the series \sum_{n=1}^{\infty} a_n an converges absolutely.If L > 1, the series \sum_{n=1}^{\infty} a_n an diverges.If L = 1, the Root Test is inconclusive, and no conclusion can be drawn.Example: Consider the series \sum_{n=1}^{\infty} \left(\frac{3}{4} \right)^n.Solution:Calculate the nth root of the absolute value of the terms:\sqrt[n]{\left| \left(\frac{3}{4} \right)^n \right|} = \sqrt[n]{\left(\frac{3}{4} \right)^n} = \frac{3}{4}Take the limit as n approaches infinity:L = \lim_{n \to \infty} \sqrt[n]{\left| \left(\frac{3}{4} \right)^n \right|} = \frac{3}{4}Since L = 3/4 < 1, the series \sum_{n=1}^{\infty} \left(\frac{3}{4} \right)^n converges absolutely.Alternating Series Test (Leibniz’s Test) Alternating Series Test, also known as Leibniz’s Test, is used to determine the convergence of an alternating series. An alternating series is one whose terms alternate in sign. Consider an alternating series of the form: \sum_{n=1}^{\infty} (-1)^{n-1} b_nThe series \sum_{n=1}^{\infty} (-1)^{n-1} b_n converges if the following two conditions are met:bn is decreasing: bn + 1 ≤ bn for all n ≥ N (for some index N).bn tends to zero: limn→∞bn = 0.Example: Consider the alternating harmonic series: \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdotsSolution:Check if bn = 1/n is decreasing: b_{n+1} = \frac{1}{n+1} \leq \frac{1}{n} = b_nCheck if bn converges to 0: \lim_{n \to \infty} \frac{1}{n} = 0Since both conditions are met, the series \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} converges.Raabe’s TestRaabe’s Test provides a more refined approach to analyzing the convergence of an infinite series by examining the behavior of the ratio of successive terms, similar to the Ratio Test but with an additional factor that can make it more sensitive to convergence. Consider a series \sum_{n=1}^{\infty}a_n. To apply Raabe’s Test, we define the limit RRR as follows:R = \lim_{n \to \infty} n \left(1 - \left| \frac{a_{n+1}}{a_n} \right|\right)The test then provides the following criteria:If R > 1, the series \sum_{n=1}^{\infty} a_n converges absolutely.If R < 1, the series \sum_{n=1}^{\infty} a_n diverges.If R = 1, the test is inconclusive, and the series may converge or diverge; other methods must be used to determine the behavior of the series.Example: Consider the series \sum_{n=1}^{\infty} \frac{n!}{n^n}.Solution:Calculate the ratio of successive terms:\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{\frac{(n+1)!}{(n+1)^{n+1}}}{\frac{n!}{n^n}} \cdot \frac{n^n}{n!} \right| = \left| \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right| = \left| \frac{(n+1)n^n}{(n+1)^{n+1}} \right| = \left| \frac{n^n}{(n+1)^n} \cdot \frac{1}{n+1} \right|Simplify the expression:\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{n^n}{(n+1)^n} \cdot \frac{1}{n+1} \right| = \left| \left(\frac{n}{n+1}\right)^n \cdot \frac{1}{n+1} \right|Take the limit of \left(1 - \left| \frac{a_{n+1}}{a_n} \right|\right):R = \lim_{n \to \infty} n \left(1 - \left(\frac{n}{n+1} \right)^n \cdot \frac{1}{n+1} \right)As n becomes very large, \left(\frac{n}{n+1} \right)^n approaches 1/e, so:R = \lim_{n \to \infty} n \left(1 - \frac{1}{e(n+1)} \right) \approx \lim_{n \to \infty} n \left(1 - \frac{1}{en} \right) = \lim_{n \to \infty} n - \frac{n}{en} = \lim_{n \to \infty} n - \frac{1}{e} = \infty Since R = ∞, which is greater than 1, the series \sum_{n=1}^{\infty} \frac{n!}{n^n} converges absolutely.Read More:Practice Problems on Convergence TestsProblem 1: Consider the geometric series \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n.Problem 2: Consider the alternating series \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.Problem 3: Consider the series \sum_{n=1}^{\infty} \frac{n!}{3^n}.Problem 4: Consider the series \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n.Problem 5: Consider the series \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3}.Problem 6: Consider the series \sum_{n=1}^{\infty} \frac{1}{n^2}.Problem 7: Consider the series\sum_{n=1}^{\infty} \frac{1}{n} \quad.Problem 8: Consider the series \sum_{n=1}^{\infty} \frac{1}{n^4}.Problem 9: Consider the series \sum_{n=1}^{\infty} \frac{n^2}{2^n}.Problem 10: Consider the series \sum_{n=1}^{\infty} \frac{n^3}{3^n}.ConclusionThe Convergence tests are essential tools in the determining whether an infinite series converges or diverges. By applying tests like the Ratio Test, Root Test Alternating Series Test and others one can assess the behavior of the series. These tests help in the analyzing the series' terms and their limits, providing the insight into the series' sum or divergence. The Mastery of the convergence tests is crucial for the solving complex problems in the calculus and advanced mathematics. Show Mobile Notice Show All Notes Hide All Notes Mobile Notice You appear to be on a device with a "narrow" screen width (i.e. you are probably on a mobile phone). Due to the nature of the mathematics on this site it is best viewed in landscape mode. If your device is not in landscape mode many of the equations will run off the side of your device (you should be able to scroll/swipe to see them) and some of the menu items will be cut off due to the narrow screen width. In the previous section we spent some time getting familiar with series and we briefly defined convergence and divergence. Before worrying about convergence and divergence of a series we wanted to make sure that we’ve started to get comfortable with the notation involved in series and some of the various manipulations of series that we will, on occasion, need to be able to do. As noted in the previous section most of what we were doing there won’t be done much in this chapter. So, it is now time to start talking about the convergence and divergence of a series as this will be a topic that we’ll be dealing with to one extent or another in almost all of the remaining sections of this chapter. So, let’s recap just what an infinite series is and what it means for a series to be convergent or divergent. We’ll start with a sequence \(\left\{ {{a_n}} \right\}_{n = 1}^\infty \) and again note that we’re starting the sequence at \(n = 1\) only for the sake of convenience and it can, in fact, be anything. Next, we define the partial sums of the series as, \[\begin{align*}&{s_1} = {a_1}\\ &{s_2} = {a_1} + {a_2}\\ &{s_3} = {a_1} + {a_2} + {a_3}\\ &{s_4} = {a_1} + {a_2} + {a_3} + {a_4}\\ &\hspace{0.25in}\, \vdots \\ &{s_n} = {a_1} + {a_2} + {a_3} + {a_4} + \cdots + {a_n} = \sum\limits_{i = 1}^n {{a_i}} \end{align*}\] and these form a new sequence, \(\left\{ {{s_n}} \right\}_{n = 1}^\infty \). An infinite series, or just series here since almost every series that we’ll be looking at will be an infinite series, is then the limit of the partial sums. Or, \[\sum\limits_{i = 1}^\infty {{a_i}} = \mathop {\lim }\limits_{n \to \infty } {s_n}\] It is important to remember that \(\sum\limits_{i = 1}^\infty {{a_i}} \) is really nothing more than a convenient notation for \(\mathop {\lim }\limits_{n \to \infty } \sum\limits_{i = 1}^n {{a_i}} \) so we do not need to keep writing the limit down. We do, however, always need to remind ourselves that we really do have a limit there! If the sequence of partial sums is a convergent sequence (i.e. its limit exists and is finite) then the series is also called convergent and in this case if \(\mathop {\lim }\limits_{n \to \infty } {s_n} = s\) then, \(\sum\limits_{i = 1}^\infty {{a_i}} = s\). Likewise, if the sequence of partial sums is a divergent sequence (i.e. its limit doesn’t exist or is plus or minus infinity) then the series is also called divergent. Let’s take a look at some series and see if we can determine if they are convergent or divergent and see if we can determine the value of any convergent series we find. Example 1 Determine if the following series is convergent or divergent. If it converges determine its value. \[\sum\limits_{n = 1}^\infty n \] Show Solution To determine if the series is convergent we first need to get our hands on a formula for the general term in the sequence of partial sums. \[{s_n} = \sum\limits_{i = 1}^n i \] This is a known series and its value can be shown to be, \[{s_n} = \sum\limits_{i = 1}^n i = \frac{{n\left({n + 1} \right)}}{2}\] Don’t worry if you didn’t know this formula (we’d be surprised if anyone knew it...) as you won’t be required to know it in my course. So, to determine if the series is convergent we will first need to see if the sequence of partial sums, \[\left\{ {\frac{{n\left({n + 1} \right)}}{2}} \right\}_{n = 1}^\infty \] is convergent or divergent. That’s not terribly difficult in this case. The limit of the sequence terms is, \[\mathop {\lim }\limits_{n \to \infty } \frac{{n\left({n + 1} \right)}}{2} = \infty \] Therefore, the sequence of partial sums diverges to \(\infty \) and so the series also diverges. So, as we saw in this example we had to know a fairly obscure formula in order to determine the convergence of this series. In general finding a formula for the general term in the sequence of partial sums is a very difficult process. In fact after the next section we’ll not be doing much with the partial sums of series due to the extreme difficulty faced in finding the general formula. This also means that we’ll not be doing much work with the value of series since in order to get the value we’ll also need to know the general formula for the partial sums. We will continue with a few more examples however, since this is technically how we determine convergence and the value of a series. Also, the remaining examples we’ll be looking at in this section will lead us to a very important fact about the convergence of series. So, let’s take a look at a couple more examples. Example 2 Determine if the following series converges or diverges. If it converges determine its sum. \[\sum\limits_{n = 2}^\infty {\frac{1}{{{n^2} - 1}}} \] Show Solution This is actually one of the few series in which we are able to determine a formula for the general term in the sequence of partial sums. However, in this section we are more interested in the general idea of convergence and divergence and so we’ll put off discussing the process for finding the formula until the next section. The general formula for the partial sums is, \[{s_n} = \sum\limits_{i = 2}^n {\frac{1}{{{i^2} - 1}}} = \frac{3}{4} - \frac{1}{{2n}} - \frac{1}{{2\left({n + 1} \right)}}\] and in this case we have, \[\mathop {\lim }\limits_{n \to \infty } {s_n} = \mathop {\lim }\limits_{n \to \infty } \left({\frac{3}{4} - \frac{1}{{2n}} - \frac{1}{{2\left({n + 1} \right)}}} \right) = \frac{3}{4}\] The sequence of partial sums converges and so the series converges also and its value is, \[\sum\limits_{n = 2}^\infty {\frac{1}{{{n^2} - 1}}} = \frac{3}{4}\] Example 3 Determine if the following series converges or diverges. If it converges determine its sum. \[\sum\limits_{n = 0}^\infty {{{\left({ - 1} \right)}^n}} \] Show Solution In this case we really don’t need a general formula for the partial sums to determine the convergence of this series. Let’s just write down the first few partial sums. \[\begin{align*}&{s_0} = 1\\ &{s_1} = 1 - 1 = 0\\ &{s_2} = 1 - 1 + 1 = 1\\ &{s_3} = 1 - 1 + 1 - 1 = 0\\ &\text{ etc.}\end{align*}\] So, it looks like the sequence of partial sums is, \[\left\{ {{s_n}} \right\}_{n = 0}^\infty = \left\{ {1,0,1,0,1,0,1,0,1, \ldots } \right\}\] and this sequence diverges since \(\mathop {\lim }\limits_{n \to \infty } {s_n}\) doesn’t exist. Therefore, the series also diverges. Example 4 Determine if the following series converges or diverges. If it converges determine its sum. \[\sum\limits_{n = 1}^\infty {\frac{1}{{{{3^n - 1}}}}} \] Show Solution Here is the general formula for the partial sums for this series. \[{s_n} = \sum\limits_{i = 1}^n {\frac{1}{{{{3^i - 1}}}}} = \frac{3}{{2}\left({1 - \frac{1}{{{{3^n}}}}} \right)\] Again, do not worry about knowing this formula. This is not something that you’ll ever be asked to know in my class. In this case the limit of the sequence of partial sums is, \[\mathop {\lim }\limits_{n \to \infty } {s_n} = \mathop {\lim }\limits_{n \to \infty } \frac{3}{2}\left({1 - \frac{1}{{{{3^n}}}}} \right) = \frac{3}{2}\] The sequence of partial sums is convergent and so the series will also be convergent. The value of the series is, \[\sum\limits_{n = 1}^\infty {\frac{1}{{{{3^n - 1}}}}} = \frac{3}{2}\] As we already noted, do not get excited about determining the general formula for the sequence of partial sums. There is only going to be one type of series where you will need to determine this formula and the process in that case isn’t too bad. In fact, you already know how to do most of the work in the process as you’ll see in the next section. So, we’ve determined the convergence of four series now. Two of the series converged and two diverged. Let’s go back and examine the series terms for each of these. For each of the series let’s take the limit as \(n\) goes to infinity of the series terms (not the partial sums!!). \[\begin{align*}& \mathop {\lim }\limits_{n \to \infty } n = \infty & \hspace{0.75in}& {\mbox{this series diverged}}\\ & \mathop {\lim }\limits_{n \to \infty } \frac{1}{{{n^2} - 1}} = 0 & \hspace{0.75in}& {\mbox{this series converged}}\\ & \mathop {\lim }\limits_{n \to \infty } {\left({ - 1} \right)^n}{\mbox{ doesn't exist}} & \hspace{0.5in}& {\mbox{this series diverged}}\\ & \mathop {\lim }\limits_{n \to \infty } \frac{1}{{{{3^n - 1}}}} = 0 & \hspace{0.75in}& {\mbox{this series converged}}\end{align*}\] Notice that for the two series that converged the series term itself was zero in the limit. This will always be true for convergent series and leads to the following theorem. Theorem If \(\sum {{a_n}} \) converges then \(\mathop {\lim }\limits_{n \to \infty } {a_n} = 0\). Proof First let’s suppose that the series starts at \(n = 1\). If it doesn’t then we can modify things as appropriate below. Then the partial sums are, \[{s_{n - 1}} = \sum\limits_{i = 1}^{n - 1} {{a_i}} = {a_1} + {a_2} + {a_3} + {a_4} + \cdots + {a_{n - 1}}\hspace{0.25in}{s_n} = \sum\limits_{i = 1}^n {{a_i}} = {a_1} + {a_2} + {a_3} + {a_4} + \cdots + {a_{n - 1}} + {a_n}\] Next, we can use these two partial sums to write, \[{a_n} = {s_n} - {s_{n - 1}}\] Now because we know that \(\sum {{a_n}} \) is convergent we also know that the sequence \(\left\{ {{s_n}} \right\}_{n = 1}^\infty \) is also convergent and that \(\mathop {\lim }\limits_{n \to \infty } {s_n} = s\) for some finite value \(s\). However, since \(n - 1 \to \infty \) as \(n \to \infty \) we also have \(\mathop {\lim }\limits_{n \to \infty } {s_{n - 1}} = s\). We now have, \[\mathop {\lim }\limits_{n \to \infty } {a_n} = \mathop {\lim }\limits_{n \to \infty } \left({{s_n} - {s_{n - 1}}} \right) = \mathop {\lim }\limits_{n \to \infty } {s_n} - \mathop {\lim }\limits_{n \to \infty } {s_{n - 1}} = s - s = 0\] Be careful to not misuse this theorem! This theorem gives us a requirement for convergence but not a guarantee of convergence. In other words, the converse is NOT true. If \(\mathop {\lim }\limits_{n \to \infty } {a_n} = 0\) the series may actually diverge! Consider the following two series. \[\sum\limits_{n = 1}^\infty {\frac{1}{{n}}} \hspace{0.75in}\hspace{0.25in}\sum\limits_{n = 1}^\infty {\frac{1}{{{{n^2}}}}} \] In both cases the series terms are zero in the limit as \(n\) goes to infinity, yet only the second series converges. The first series diverges. It will be a couple of sections before we can prove this, so at this point please believe this and know that you’ll be able to prove the convergence of these two series in a couple of sections. Again, as noted above, all this theorem does is give us a requirement for a series to converge. In order for a series to converge the series terms must go to zero in the limit. If the series terms do not go to zero in the limit then there is no way the series can converge since this would violate the theorem. This leads us to the first of many tests for the convergence/divergence of a series that we’ll be seeing in this chapter. Divergence Test If \(\mathop {\lim }\limits_{n \to \infty } {a_n} e 0\) then \(\sum {{a_n}} \)will diverge. Again, do NOT misuse this test. This test only says that a series is guaranteed to diverge if the series terms don’t go to zero in the limit. If the series terms do happen to go to zero the series may or may not converge! Again, recall the following two series, \[\begin{align*} &\sum\limits_{n = 1}^\infty {